Chapter 6

Finance 300

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Learning points

- What if we add multiple cash flows to our timeline?
- How loans are structured?
- How interest rates are quoted?

Valuing a Stream of (Un)Even Cash Flows

- We compute the present value of this cash flow in two steps:
  - First, compute the present value of *each* cash flow.
  - Second, **combine** the present values.

$$PV = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_N}{(1+r)^N}$$

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Valuing a Stream of Even Cash Flows

- You think you will be able to deposit $4,000 at the end of each of the next three years into an account earning 8% interest. You currently have $7,000? How much will you have in three years? Four years? Twenty years?

Valuing a Stream of Even Cash Flows

- You are offered an investment that pays 1,000 at the end of each year for the next 4 years. The discount rate is 4%. What is the maximum you would be willing to pay for this investment.

Valuing a Stream of Uneven Cash Flows - Example 1

- You are considering an investment that will pay you $1,000 in one year, $2,000 in two years and $3,000 in three years.
- If you want to earn 10% per year on your money, how much would you be willing to pay (today) for this investment?
- Hint: You need to calculate the PV of the 3-year cash flow.

Hockey Example

		Annual Salary	PV	Annual Salary	PV	Annual Salary	PV	Annual Salary	PV
2015-16	0								
2016-17	1	$14,000,000	$12,727,273	$13,800,000	$12,545,455	$10,500,000	$9,545,455	$6,900,000	$6,272,727
2017-18	2	$13,000,000	$10,743,802	$13,800,000	$11,404,959	$10,500,000	$8,677,686	$6,900,000	$5,702,479
2018-19	3	$12,000,000	$9,015,778	$13,800,000	$10,368,144	$10,500,000	$7,888,805	$7,000,000	$5,259,204
2019-20	4	$11,000,000	$7,513,148	$12,000,000	$8,196,161	$10,500,000	$7,171,641	$9,800,000	$6,693,532
2020-21	5	$8,000,000	$4,967,371	$9,800,000	$6,085,029	$10,500,000	$6,519,674	$12,000,000	$7,451,056
2021-22	6	$8,000,000	$4,515,791	$7,000,000	$3,951,318	$10,500,000	$5,926,976	$13,800,000	$7,789,740
2022-23	7	$7,000,000	$3,592,107	$6,900,000	$3,540,791	$10,500,000	$5,388,160	$13,800,000	$7,081,582
2023-24	8	$7,000,000	$3,265,552	$6,900,000	$3,218,901	$10,500,000	$4,898,327	$13,800,000	$6,437,802
Contract Value		**$80,000,000**		**$84,000,000**		**$84,000,000**		**$84,000,000**	
Present value		**$56,340,820.51**		**$59,310,757.36**		**$56,016,725.08**		**$52,688,122.07**	

Annuities and **Perpetuities** Defined

- **Annuity** – finite series of *equal* payments that occur at regular intervals
  - If the first payment occurs at the end of the period, it is called an **ordinary annuity**
  - If the first payment occurs at the beginning of the period, it is called an **annuity due**
- **Perpetuity**: infinite series of *equal* payments

Annuities and Perpetuities: Basic Formulas

Perpetuity: **PV = C / r**

– You need to memorize the perpetuity formula!

– Annuities:

$$PV = C\left[\frac{1-\frac{1}{(1+r)^t}}{r}\right]$$

$$FV = C\left[\frac{(1+r)^t - 1}{r}\right]$$

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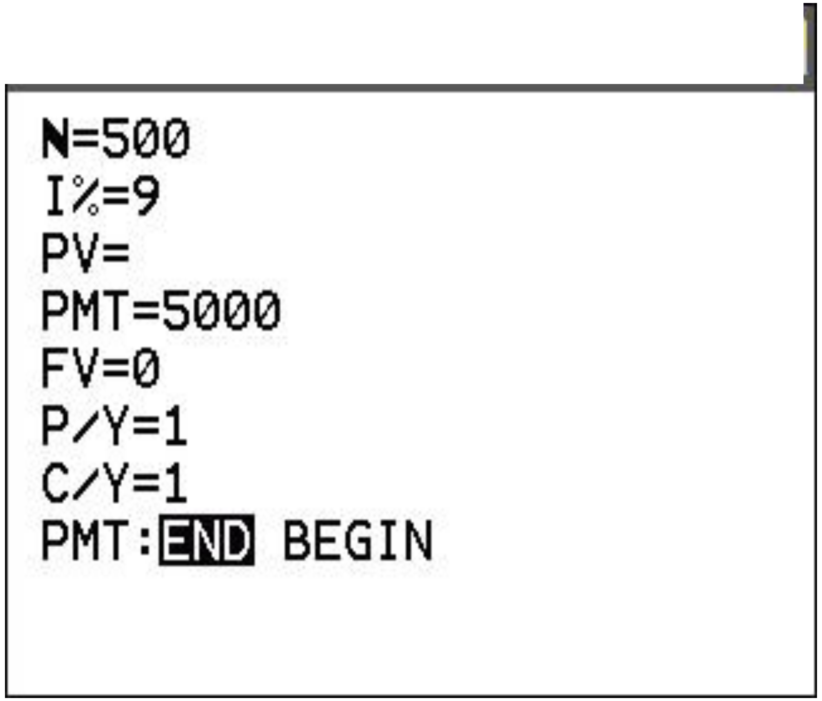
Two types of Annuities and the Calculator Settings

- Ordinary Annuity versus Annuity Due
  - Annuity due: first payment made right now (time 0)
  - Ordinary Annuity: first payment made 1 period from now (time 1)
  - **Most problems are ordinary annuities**
  - You can switch the setting:
    - "END" highlighted for PMT => ordinary annuity
    - "BEGIN" highlighted for PMT=> annuity due
  - Note : Annuity Due = Ordinary Annuity*(1+r)



Ordinary Annuity Example 1: Finding PV

- Suppose you win the Publishers Clearinghouse $10 million sweepstakes. The money is paid in equal annual installments of $333,333.33 over 30 years.
- If the appropriate discount rate is 5%, how much is the sweepstakes actually worth today?

Ordinary Annuity Example 2: Finding the Payment

- Suppose you want to borrow $20,000 for a new car.
- You can borrow at 8% per year, compounded monthly (8%/12 = .66667% per month).
- If you take a 4-*year* loan, what is your monthly payment?
- Answer: *Make everything consistent with the p* *frequency!*

$$PV = C\left[\frac{1-\frac{1}{(1+r)^t}}{r}\right]$$

Calculator:

N=4x12=48, I=0.66667, PV=20000; PMT = ? = $488.26

Ordinary Annuity Example 3: Finding the Interest Rate

- Suppose you borrow $10,000 from your parents to buy a car.
- You agree to pay $672.15 per *quarter* for 5 years.
- What is the *quarterly* interest rate?
- **Answer**:
  - N =
  - PV =
  - PMT =
  - I% = ? =

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Calculator: How do I know the sign?

Cash outflows are negative and inflows are positive.

- You borrow 10,000 and make payments of 940 a year for 15 years. What is the rate?
  - PV=+10,000 and PMT=-940
- There is an investment that pays 300 a year for 3 years how much are you willing to pay for it today? 5% interest.
- PMT=+300 and PV=-ANSWER

Note: We can switch these signs and still get the same answers. They indicate point of view.

Ordinary Annuity Example 4: Future Values for Annuities

- Suppose you begin saving for your retirement by depositing $2,000 per year in an IRA (*Individual Retirement Account)*
- If the (annual) interest rate is 7.5%, how much will you have in 40 years?

Answer:

Annuity Due Example: Finding FV

You are saving for a new house, and you put $10,000 per year in an account paying 8% per year. The first payment is made *today*.

How much will you have at the END of 3 years?

Key to ordinary annuity: FV occurs at the same time as the last payment!

If these cash flows were ordinary annuity, when will FV happen and how much?

Perpetuity Example: Finding the Payment

- You just won the lottery, and you want to endow a professorship at your *alma mater*.
- You are willing to donate **$4 million** of your winnings for this purpose.
- If the university earns **5%** per year on its investments, and the professor will be receiving her first payment in one year, *how much will the endowment pay her each year?*

Growing Perpetuity: Concept and Formula

A ***growing*** perpetuity is a ***growing*** stream of cash flows that lasts **forever**.

$$PV = \frac{C}{(1+r)} + \frac{C \times (1+g)}{(1+r)^2} + \frac{C \times (1+g)^2}{(1+r)^3} + \cdots$$

$$PV = \frac{C}{r-g}$$

In order for the formula to make sense, $g < r$

C occurs one year from today

Growing Perpetuity Example: Finding Payment (with a twist)

- In the previous example of endowment, you planned to donate $4 million to your *alma mater* to fund an endowed professorship. Given an annual return of 5%, the professor receiving the endowment would be able to receive $200,000 *per year* from your generosity.
- Now we take into account the inflation rate. **Assume the inflation rate is 2% per year.**
- How much can a professor be paid *in the first year* to allow her annual salary to *increase by 2%* each *year (forever)* and keep the principal intact?
- How much can she receive ***in the second year***?

Growing Annuity

- A **growing annuity is** a stream of *t* growing cash flows paid at regular intervals.
- A growing annuity is essentially a growing perpetuity *that eventually comes to an end*.

$$PV = \frac{C}{r-g}\left[1-\left(\frac{(1+g)}{(1+r)}\right)^{t}\right]$$

$$PV = C\left[\frac{1-\left(\frac{(1+g)}{(1+r)}\right)^{t}}{r-g}\right]$$

- The following timeline shows a growing annuity with *initial cash flow C* growing at a rate of *g* every period until period *t* :

C $C(1+g)$ $C(1+g)^2$ $C(1+g)^{t-1}$

Note:

(1) By convention, first cash flow happens at date 1 and does *not* grow.

(2) For growing *annuities*, g > r *or* g < r. (Growing *perpetuities* assume g < r.)

Growing Annuity Example: Finding the PV

- Your uncle has left you an estate:
  - Next year, you will receive $10,000
  - The following year, you will receive $11,000
  - The following year, you will receive $12,100
  - And so on, to the end of 20^{th} year, *at a constant growth rate of 10%*.
- You'd rather have the money right now; *For what price would you sell your inheritance if the interest rate is 8% per year*? => Think 8% as the investment return you could get in the future get if you had the money *now*.
- *Note:* ***You will NOT be able to get to the answer using a financial calculator. You have to use the formula and basic algebra (when taking the exams).***

Growing Annuity: An Example - Answer

- Timeline:

0	1	2	3	…	20
r=8%					
	10	10(1+.1)	$10(1+.1)^2$	…	$10(1+.1)^{19}$

$$PV = C\left[\frac{1-\left(\frac{(1+g)}{(1+r)}\right)^t}{r-g}\right] = 10{,}000\left(\frac{1-\left(\frac{1+.1}{1+.08}\right)^{20}}{0.08-0.1}\right) = 221{,}686.5$$

Note that r<g in this case.

Interest Rate Quotes and Adjustments

- In this section, we clarify what is really meant by “the interest rates” and how to find the appropriate rate for determining the time value of money.
- Interest rates are the <u>prices</u> of either receiving money or paying out money over time.
- In practice,
  - Banks typically compound interest quarterly, monthly or daily
  - Mortgage companies typically compound interest monthly
- To make things easily comparable, interest rates are usually expressed <u>annually.</u>
- It turns out the rate that is perhaps most familiar to people, **the APR, is <u>NOT</u> the one that should be used in investment decisions.**

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Effective Annual Rate (EAR)

- **EAR** is the actual rate paid or received *after accounting for compounding that occurs during the year*
- If you want to compare two alternative investments with different compounding periods on an annual basis, compute the EAR and use that for comparison. => Remember!

Effective Annual Rate (EAR)

- Quoted rate is 8% (stated/quoted rate) semi-annually? Therefore 4% every 6 months? Is 4% every 6 months the same as 8% a year? Let's check (take 100 invested for one year):

Annual Percentage Rate (APR)

- APR is the most common way of quoting the interest rate. It is simple to calculate.
- APR is NOT the actual price of money because it *ignores* compounding interests. Hence APR is typically lower than EAR.
- ***By definition***, **APR = periodic rate x m, where m = Number of compounding periods/year.** In other words, **Periodic Rate = APR / m**
- **Example 1: What is the APR if the monthly rate is .5%?**
- **Example 2: What is the APR if the *semiannual* rate is .5%?**

NOTE: APR IS THE QUOTED OR STATED RATE.

Rule of Thumb When Computing Interest Rates

- ALWAYS make sure that the interest rate and the compounding period match.
  - If you are looking at daily periods, you need a daily rate.
  - If you are looking at monthly periods, you need a monthly rate.
- NEVER divide the EAR by the number of periods per year.
  - Doing so will **NOT** give you the **correct** period rate.

APR, EAR and Period Rate in ONE Formula

$$\text{EAR} = \left[1 + \frac{\text{APR}}{\text{m}}\right]^{\text{m}} - 1$$

- The APR is the "quoted rate"
- *m* is the number of compounding periods per year
- **EAR is the *ACTUAL* annual rate.**

Example 1: Calculating EAR When Given APR

- You are looking at two savings accounts. First account pays 5.25% APR with ***daily*** compounding. The second one pays 5.3% APR with ***semiannual*** compounding. **Which account should you choose**?
- Answer:

$$EAR = \left[1 + \frac{APR}{m}\right]^m - 1$$

.

How to Calculate APR When Given EAR?

- If you know the EAR, how would you compute the APR?
- Answer: Rearrange the EAR equation and you get:

$$APR = m\left[(1 + EAR)^{\frac{1}{m}} - 1\right]$$

$$EAR = \left[1 + \frac{APR}{m}\right]^m - 1$$

$$\left[1 + \frac{APR}{m}\right]^m = EAR + 1$$

$$1 + \frac{APR}{m} = (EAR + 1)^{1/m}$$

$$\frac{APR}{m} = (EAR + 1)^{1/m} - 1$$

$$APR = m[(EAR + 1)^{1/m} - 1]$$

Computing from EAR to APR: Example

- Suppose you want to earn an **effective** rate of 12% and you are looking at an account that compounds on a **monthly** basis.
- What is the APR? (Keep *at least four decimal places* in the answer.)
- Answer:

$$APR = m\left[(1+EAR)^{1/m} - 1\right]$$

- Concept check 1: APR < EAR when m ≥ 1.
- Concept check 1: APR = EAR when m = 1.

Example2: Computing Loan Payments Given APRs

- Suppose you want to buy a new computer system and the store is willing to allow you to make **monthly** payments.
- The computer system costs $3,500 (today).
- The loan period is for 2 **years**, and the interest rate is 16.9% APR with **monthly** compounding.
- What is your **monthly** payment?
- Answer:

Credit card APRs

Continuous Compounding

- Sometimes investments or loans are based on continuous compounding
- **EAR = $e^q - 1$**
- The *e* is the **exponential function**: *e* = 2.71828…
  - It can be found on your calculator (normally denoted by e^x) and in Excel (*exp(x)*).
- Example: What is the EAR of 7% APR compounded continuously? (Keep four decimal places)

- **APR = *ln* (1+EAR)**

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Compounding

Compounding Period	Number of Times Compounded	Effective Annual Rate
Year	1	10.00000%
Quarter	4	10.38129
Month	12	10.47131
Week	52	10.50648
Day	365	10.51558
Hour	8,760	10.51703
Minute	525,600	10.51709

Continuous Compounding Example

- Suppose the continuously compounded rate this year is 6% and it will be 7% next year. If you invest $100 today, how much will you have in 2 years? (Keep 2 decimal places in final answer)
- Algebra reminder:

$$e^x e^y = e^{x+y}$$

- FV = \$100 ($e^{.06}$) ($e^{.07}$) = <u>\$100 ($e^{.06+.07}$) = \$100 ($e^{.13}$) = \$113.88</u>

Loan Types

1. Pure Discount loans
   - Pay a single lump sum at the end of the loan
2. Interest-only loans
   - Borrower pays interest each period and repay principal at some point in the future.
   - Ex. Bonds (next chapter)
3. Amortized loans
   - Borrower makes regular principal reductions
   - Ex. Car loans, Mortgage, Student loans, Credit cards.

Pure Discount Example

A borrower is able to repay $25,000 in five years. We as the lender require a 12% interest rate. How much would we be willing to lend today.

Interest-only

- Covered extensively in Chapter 7

Amortized

Suppose you buy a house and your mortgage is $120,000 for 15 years at 6%. (Assume equal payments). What is the **monthly** payment? What does the amortization schedule look like?